

# NONWANDERING SETS OF INTERVAL SKEW PRODUCTS

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**ABSTRACT.** In this paper we consider a class of skew products over transitive subshifts of finite type with interval fibers. For a natural class of 1-parameter families we prove that for all but countably many parameter values the nonwandering set (in particular, the union of all attractors and repellers) has zero measure. As a consequence, the same holds for a residual subset of the space of skew products.

## 1. INTRODUCTION

Skew products over hyperbolic dynamics or, equivalently, over subshifts of finite type is quite a standard component of the modern theory of partially hyperbolic dynamical systems. A detailed review of this role of skew products can be found, for instance, in our thorough 3-pages introduction to [2].

In [2] which is the prequel to this paper, we started quite an ambitious project to understand such skew products in the case when the fibers are intervals of real line, and the fiber maps are orientation preserving diffeomorphisms. We gave a complete description of possible dynamics for so-called *step* skew products: such that the dependence of fiber maps on the base coordinate is piecewise constant.

In this sequel we manage to relax the *step* condition and to extend some of the results of [2] to *all* skew products. Namely, in Theorem 4.3 we show that for a residual set of such skew products, the nonwandering set (including all the attractors and repellers) has zero standard measure which is the product of the ergodic invariant measure of the subshift in the base and the Lebesgue measure in the fiber.

For this, we prove a stronger statement which is Theorem 4.1: for any monotone 1-parameter family, for all but countably many parameter values the standard measure of so-called *anchored* set (which includes the nonwandering set) is zero.

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## 2. NOTATIONS

Let  $\sigma: \Sigma \rightarrow \Sigma$  be a transitive subshift of finite type (a topological Markov chain),  $\Sigma \subset \{1, \dots, N\}^{\mathbb{Z}}$ . We endow  $\Sigma$  with a metric defined by the formula

$$(2.1) \quad d(\bar{\omega}, \omega) = \begin{cases} 2^{-\min\{|n|: \bar{\omega}_n \neq \omega_n\}}, & \bar{\omega} \neq \omega, \\ 0, & \bar{\omega} = \omega, \end{cases} \quad \bar{\omega}, \omega \in \Sigma.$$

Fix any  $\sigma$ -invariant ergodic measure  $\mu$  on  $\Sigma$ .

Let  $I \subset \mathbb{R}$  be a unit interval. In this paper, we will be considering the following class  $\mathcal{S}$  of skew products  $F: \Sigma \times I \rightarrow \Sigma \times I$ :

- $F: (\omega, x) \mapsto (\sigma\omega, f_\omega(x))$ ;
- the maps  $f_\omega: I \rightarrow f_\omega(I)$  are orientation preserving  $C^1$  diffeomorphisms which send  $I$  strictly inside itself;
- the map  $f_{(\cdot)}: \Sigma \rightarrow \text{Diff}^1(I)$  is continuous.

We equip  $\mathcal{S}$  with a metric as follows:

$$\text{dist}(F, \tilde{F}) = \max_{\omega} \text{dist}_{C^1}(f_\omega^{\pm 1}, \tilde{f}_\omega^{\pm 1}).$$

The *standard measure*  $\mathbf{m}$  on  $\Sigma \times I$  is the product of Markov measure  $\mu$  in the base and the Lebesgue measure in the fiber.

## 3. DRIFTING AND ANCHORED REGIONS

Following [2], we define drifting graphs and points. Let  $\varphi_i: \Sigma \rightarrow I$  be two arbitrary functions,  $\Gamma_i$  be their graphs,  $i = 1, 2$ .

**Definition 3.1.** We write  $\varphi_1 < \varphi_2$  whenever for any  $\omega \in \Sigma$

$$\varphi_1(\omega) < \varphi_2(\omega).$$

We also write  $\Gamma_1 < \Gamma_2$  in this case.

A general property of any skew product is that the image  $F(\Gamma)$  of any graph  $\Gamma$  is also a graph of some function.

**Definition 3.2.** We say that a graph  $\Gamma$  *drifts up (down)* if  $F(\Gamma) > \Gamma$  (respectively,  $F(\Gamma) < \Gamma$ ).

Recall that we assume the fiber maps  $f_\omega$  to be monotone increasing. Thus  $F(\Gamma) > \Gamma$  implies  $F^{n+1}(\Gamma) > F^n(\Gamma)$  for any  $n \in \mathbb{Z}$ , and  $F(\Gamma) < \Gamma$  implies  $F^{n+1}(\Gamma) < F^n(\Gamma)$  for any  $n \in \mathbb{Z}$ , provided the backward iterates are well-defined. The strip between  $F^n(\Gamma)$  and  $F^{n+1}(\Gamma)$  is a fundamental domain for  $F$ .

**Definition 3.3.** A point  $p = (\omega, x) \in \Sigma \times I$  *drifts up* if there exists a continuous map  $\gamma: \Sigma \rightarrow I$  with the graph  $\Gamma$  such that

- $\Gamma$  drifts up under  $F$ ;
- the point  $p$  is between  $\Gamma$  and its image:

$$\gamma(\omega) < x < f_{\sigma^{-1}\omega}(\gamma(\sigma^{-1}\omega)),$$

Denote the set of the points drifting up in  $F$  by  $\Delta(F)$ . In the same way we define the set of points *drifting down* which we denote by  $\nabla(F)$ . Finally, denote by  $\square(F) = (\Sigma \times I) \setminus (\Delta(F) \cup \nabla(F))$  the *anchored set* which is the complement to the points drifting up or down.

*Remark 3.4.* For the nonwandering set  $\Omega(F)$  we always have  $\Omega(F) \subset \square(F)$  because the sets  $\Delta(F)$  and  $\nabla(F)$  are open and disjoint from  $\Omega(F)$ .

**Proposition 3.5.** *For any  $F$ , we have  $\Delta(F) \cap \nabla(F) = \emptyset$ .*

*Proof.* Assume that  $\Delta(F) \cap \nabla(F) \neq \emptyset$  for some skew product  $F$ . Because the set  $\Delta(F) \cap \nabla(F)$  is open and the periodic points are dense in  $\Sigma$ , we can pick  $p = (\omega, x) \in \Delta(F) \cap \nabla(F)$  such that  $\omega$  is periodic. Let  $n$  be its minimal period. Then  $F^n(p) = (\bar{x}, \bar{\omega})$  belongs to the same fiber as  $p$ . Note that if  $\bar{x} > x$ , then  $p$  cannot be drifting down, and if  $\bar{x} < x$ , then  $p$  cannot be drifting up. Thus  $p \notin \Delta(F) \cap \nabla(F)$ , and the Proposition is proven.  $\square$

Now we introduce a partial ordering on the set of mild skew products with an interval fiber:

**Definition 3.6.**  $F \prec \tilde{F}$  whenever for any  $\omega, x$  we have  $f_\omega(x) < \tilde{f}_\omega(x)$ .

**Proposition 3.7.** *If  $F \prec \tilde{F}$ , then*

$$\Delta(F) \subset \Delta(\tilde{F}) \text{ and } \nabla(F) \supset \nabla(\tilde{F}).$$

*Proof.* Indeed, for any  $p \in \Delta(F)$  we can take  $\Gamma$  that satisfies Definition 3.3. Because  $F \prec \tilde{F}$ , we have  $F(\Gamma) < \tilde{F}(\Gamma)$ . Thus the same  $\Gamma$  is also valid to show  $p \in \Delta(\tilde{F})$ .

The second inclusion is proved in the same way.  $\square$

**Definition 3.8.** A family  $F_\tau$ ,  $\tau \in (\tau_1, \tau_2)$ , of skew products is *monotone increasing* if for any  $\tau_1 < \tau_2$  the skew products  $F_{\tau_1}$  and  $F_{\tau_2}$  are comparable and  $F_{\tau_1} \prec F_{\tau_2}$ .

#### 4. MAIN RESULTS

**Theorem 4.1.** *Let  $F_\tau$  be a monotone family, continuous in  $\tau$ . Then for every  $\tau$ , except for at most countable set of them, we have*

$$(4.1) \quad \mathbf{m}(\square(F_\tau)) = 0,$$

where  $\mathbf{m}$  is the standard measure. In particular, the standard measure of the nonwandering set  $\Omega(F_\tau) \subset \square(F_\tau)$  is zero.

Moreover, for any  $\varepsilon > 0$  the set  $\{\tau \mid \mathbf{m}(\square(F_\tau)) \geq \varepsilon\}$  is finite.

**Definition 4.2.** We say that a subset of the space of skew products is *small* if it is closed, nowhere dense, and any continuous monotone family intersects it at a finite number of points.

**Theorem 4.3.** *The set  $\mathcal{K} = \{F \mid \mathbf{m}(\square(F)) > 0\}$  is a subset of a countable union of small sets. In particular,  $\mathcal{K}$  is meager.*

Combined with Remark 3.4, this implies that a generic skew product from  $\mathcal{S}$  has a nonwandering set of zero standard measure.

In the spirit of the Large Deviations Lemma [1, Lemma 6] and the Special Ergodic Theorem [1, Theorem 6] which give an estimate of the Hausdorff dimension of “bad” sets in some partially hyperbolic systems, we conjecture the following generalization of Theorem 4.1.

**Conjecture 4.4.** *Let  $F_\tau$  be a monotone family, continuous in  $\tau$ . Then for every  $\tau$ , except for at most countable set of them, the Hausdorff dimension of  $\square(F_\tau)$  is less than the full dimension of the phase space.*

First we show that Theorem 4.1 implies Theorem 4.3.

*Proof of Theorem 4.3.* Obviously,  $\mathcal{K} = \cup_{n \in \mathbb{N}} \mathcal{K}_{1/n}$ , where

$$\mathcal{K}_\varepsilon := \{F \mid \mathbf{m}(\square(F)) \geq \varepsilon\}.$$

Let us show that for any  $\varepsilon > 0$  the set  $\mathcal{K}_\varepsilon$  is small. First of all, Theorem 4.1 implies that any monotone family intersects  $\mathcal{K}_\varepsilon$  at a finite number of points.

**Proposition 4.5.** *For any  $\varepsilon > 0$ , the set  $\mathcal{K}_\varepsilon$  is closed.*

*Proof.* Note that the set of pairs  $\mathcal{S}_\Delta = \{(p, F) \mid p \in \Delta(F)\}$  is an open subset of the Cartesian product of  $\Sigma \times I$  and the space of dynamical systems on it. The same is true for  $\mathcal{S}_\nabla$ . Thus the set

$$\mathcal{S}_\square := \{(p, F) \mid p \in \square(F)\} = \{(p, F) \mid p \notin \Delta(F) \cup \nabla(F)\}$$

is closed.

Now take any sequence of systems  $F_{(n)} \in \mathcal{K}_\varepsilon$  which converges to  $F$ . Assume that  $F \notin \mathcal{K}_\varepsilon$  which means  $\mathbf{m}(\square(F)) < \varepsilon$ . Take an open cover  $U$  of the set  $\square(F)$  such that  $\mathbf{m}(U) < \varepsilon$ . Because  $F_{(n)} \in \mathcal{K}_\varepsilon$ , we have  $\mathbf{m}(\square(F_{(n)})) \geq \varepsilon$  for all  $n \in \mathbb{N}$ . Thus  $\square(F_{(n)}) \not\subset U$ . Then for any  $n \in \mathbb{N}$  there exists a point  $p_n \in (\Sigma \times I) \setminus U$ ,  $p_n \in \square(F_{(n)})$ .

Because the set  $(\Sigma \times I) \setminus U$  is compact, we can extract from  $p_n$  a converging subsequence  $p_{n_m}$ . Then the subsequence  $(p_{n_m}, F_{(n_m)}) \in \mathcal{S}_\square$  converges to some  $(\tilde{p}, F)$ , and  $\tilde{p} \notin U$ . But because the set  $\mathcal{S}_\square$  is closed, we must have  $p \in \square(F)$ . The contradiction with  $\square(F) \subset U$  proves the Proposition.  $\square$

Because any monotone family intersects  $\mathcal{K}_\varepsilon$  at a finite number of points, the set  $\mathcal{K}_\varepsilon$  has empty interior. Thus it is nowhere dense. Theorem 4.3 is proven.  $\square$

## 5. PROOF ON THEOREM 4.1

**Definition 5.1.** A skew product  $F: \Sigma \times M \rightarrow \Sigma \times M$  is *multistep* if the fiber maps  $f_\omega$  depend only on finitely many positions in  $\omega$ .

Denote the set of skew products in  $\mathcal{S}$  depending on  $m$  positions by  $\mathcal{S}(m)$ . In the same way every continuous function can be approximated by piecewise constant functions in sup-norm, we can  $C^0(\Sigma, C^1(I))$ -approximate skew products from  $\mathcal{S}$  by multistep skew products. These approximations can be chosen to be generic in the sense of [2]. It immediately follows from [2, Theorem 2.15] that for any generic multistep skew product  $G$  we have  $\mathbf{m}(\square(G)) = 0$ .

**Proposition 5.2.** *Let  $F_1 \prec F_2$ . Then  $\mathbf{m}(\nabla(F_1) \cup \Delta(F_2)) = 1$ .*

*Proof.* Take a generic multistep skew product  $G$  such that  $F_1 \prec G \prec F_2$ . Because of the above remark,

$$\mathbf{m}(\nabla(G) \cup \Delta(G)) = 1.$$

But  $\nabla(F_1) \supset \nabla(G)$  and  $\Delta(F_2) \supset \Delta(G)$ . Thus  $\nabla(G) \cup \Delta(G) \subset \nabla(F_1) \cup \Delta(F_2)$  which proves the Proposition.  $\square$

Now fix any monotone increasing family  $F_\tau$ . Denote for brevity  $\Delta_\tau := \Delta(F_\tau)$ ,  $\nabla_\tau := \nabla(F_\tau)$ . Proposition 3.7 implies that for any  $\tau_1 < \tau_2$  we have  $\Delta_{\tau_1} \subset \Delta_{\tau_2}$  and  $\nabla_{\tau_1} \supset \nabla_{\tau_2}$ . Also denote

$$\Delta_{\tau+} := \bigcap_{\delta>0} \Delta_{\tau+\delta}, \quad \nabla_{\tau-} := \bigcap_{\delta>0} \nabla_{\tau-\delta}.$$

Obviously,  $\Delta_\tau \subset \Delta_{\tau+}$  and  $\nabla_\tau \subset \nabla_{\tau-}$ .

**Proposition 5.3.** *For any  $\tau$ ,*

- i)  $\Delta_{\tau+} \cap \nabla_\tau = \emptyset$ .
- ii)  $\mathbf{m}(\Delta_{\tau+} \cup \nabla_\tau) = 1$ .

*Proof.* Because  $F_\tau$  is continuous in  $\tau$ , for any point  $p$  the set of parameters  $\{\tau \mid p \in \nabla_\tau\}$  is open. Thus for any  $p \in \nabla_\tau$  for any small enough  $\delta > 0$  we have  $p \in \nabla_{\tau+\delta}$ . By Proposition 3.5, this implies  $p \notin \Delta_{\tau+\delta}$ . Taking the intersection over all  $\delta > 0$ , we have that  $p$  does not belong to  $\Delta_{\tau+}$ . Thus  $\Delta_{\tau+} \cap \nabla_\tau = \emptyset$ .

Let us now prove ii. By Proposition 5.2, for any  $\delta > 0$

$$\mathbf{m}\left(\nabla_\tau \bigcup \Delta_{\tau+\delta}\right) = 1.$$

By Proposition 3.7, the sets  $\Delta_{\tau+\delta}$  are monotone increasing in  $\delta$ . Take the intersection over all  $\delta > 0$  to get

$$\mathbf{m}\left(\bigcap_{\delta>0} (\nabla_\tau \cup \Delta_{\tau+\delta})\right) = 1.$$

Now factor out the term  $\nabla_\tau$  to get the required  $\mathbf{m}(\nabla_\tau \cup \Delta_{\tau+}) = 1$ .  $\square$

Now we are ready to complete the

*Proof of Theorem 4.1.* Because  $\nabla_\tau \cap \Delta_{\tau+} = \emptyset$  and  $\mathbf{m}(\nabla_\tau \cup \Delta_{\tau+}) = 1$ , we have  $\mathbf{m}(\square(F_\tau)) = \mathbf{m}((\Sigma \times [0, 1]) \setminus (\nabla_\tau \sqcup \Delta_\tau)) = \mathbf{m}((\nabla_\tau \sqcup \Delta_{\tau+}) \setminus (\nabla_\tau \sqcup \Delta_\tau)) = \mathbf{m}(\Delta_{\tau+} \setminus \Delta_\tau)$ . This can be rewritten as

$$\mathbf{m}(\Delta_{\tau+} \setminus \Delta_\tau) = \mathbf{m}\left(\bigcap_{\delta>0} \Delta_{\tau+\delta}\right) - \mathbf{m}(\Delta_\tau) = \lim_{\delta \rightarrow +0} \mathbf{m}(\Delta_{\tau+\delta}) - \mathbf{m}(\Delta_\tau).$$

So  $\mathbf{m}(\square(F_\tau))$  equals to the value of a gap of the monotone increasing function  $\mu(t) = \mathbf{m}(\Delta_t)$  at the point  $t = \tau$ . Any monotone function has at most countable number of gaps. Moreover, if the function is bounded, then for any fixed  $\varepsilon > 0$  only finitely many of the gaps can be bigger than  $\varepsilon$ . Theorem 4.1 is proven.  $\square$

Finally, we remark that all continuous monotone increasing families  $F_\tau, G_\rho$  such that  $F_0 = G_0$  are equivalent in the following sense: for any  $\tau > 0$  there exists  $\rho > 0$  such that

$$F_0 \prec G_\rho \prec F_\tau.$$

Indeed, one can take  $0 < \varepsilon = \inf_{\omega, x} (F_\tau(\omega, x) - F_0(\omega, x))$ , and take  $\rho > 0$  such that  $0 < G_\rho - G_0 < \varepsilon$ . Then  $F_0 = G_0 \prec G_\rho \prec F_\tau$ . Because this argument is symmetric with respect to switching  $F$  and  $G$ , the sets  $\Delta_{0+}, \nabla_{0-}$  depend only on  $F_0$  but not on the choice of a continuous monotone increasing family passing through  $F_0$ . Thus we could denote them just by  $\Delta^+(F_0), \nabla^-(F_0)$ .

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